2012-01

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Banik, S, Kibria, BMG (2012). On some parametric confidence intervals for the mean difference of two population. Independent Business Revew, 5(1), 11-26

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On Some Parametric Confidence Intervals for the Mean Difference of Two Populations

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ABSTRACT

We evaluated several parametric estimators for estimating the difference in means of two populations. Estimators include the ordinary-t, two versions proposed by Welch (1938) and Satterthwaite (1946), three versions proposed by Zhou and Dinh (2005), Johnson and Hall. A simulation study has been made to compare the performance of the selected estimators. Some real life business examples have been considered to illustrate the application of the methods. Based on our findings, some possible good parametric interval estimators have been proposed for future researchers, applied workers and other professionals.

Keywords: Average Width, Coverage Probability, Chi-square Distribution, Interval Estimator, Simulation Study, Skewed Population

INTRODUCTION

Skewed data are commonly used in various fields of modeling such as health science (Baklizi & Kibria, 2009; Banik & Kibria, 2010; Zhou et al., 2001), environmental science (Mudelsee & Alkio, 2007), biological science (Andersson, 2004; Gregoire & Schabenberger, 1999), engineering science and others. There is often an interest of the researcher for making inference about the differences of important measures, such as, location, scale, skeweness, kurtosis and other parameters for two independent populations. This inference can be made by constructing a confidence interval (CI) or hypothesis testing about a population parameter. Confidence interval is an interval estimate that will capture the true parameter value in the repeated samples. A convenient way to perform significance test is to compute a confidence interval for the parameter and accept the alternative hypothesis if the assumed parameter lie outside the confidence interval. In this paper, our interest is to consider the problem of comparing two independent groups in terms of measure of location. Generally, people use the normal theory to construct confidence interval for making inferences about the difference in means of two independent populations. However, in practice, the normality assumption may not be appropriate for a lot of real data. For example, a lot of health related data are skewed (Baklizi & Kibria, 2009; Zhou et al., 2001). Confidence intervals based on ordinal-t statistic suffer when samples come from skewed populations. There are several methods readily available to overcome this problem. Some of them are based on correcting the studentized t-statistic with higher order terms. Some of used the bootstrap technique. Recently Baklizi and Kibria (2009) propose a two-sample confidence interval based on the concept, median describes the center of the distribution best. Since several researchers proposed several confidence intervals for the difference of two means at several times and under different simulation conditions, their comparisons are not comparable as a while. In this paper, we evaluate several existing parametric techniques and compare them under the same simulation conditions in terms of the attainment of the nominal values of confidence intervals.

The organization the paper is as follows: The considered interval estimators are described in section 2. A simulation study along with results is discussed in section 3. Real life data are analyzed in section 4. Finally, some concluding remarks are given in section 5.

STATISTICAL METHODOLOGY

A brief description of the considered estimators is given in this section. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two independent random samples from two populations with means μ_1 and μ_2 respectively. We want to construct confidence interval for the mean difference of μ_1-μ_2. Let $\bar{x}$ and $\bar{y}$ are the sample means and s_1^2 and s_2^2 are the sample variances.

Ordinary t Method

For small samples, the commonly used t-based confidence interval for μ_1-μ_2 is defined as

$$\mathrm{LCL} = (\bar{x} - \bar{y}) - t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \text{ and } \mathrm{UCL} = (\bar{x} - \bar{y}) + t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

For large samples, the corresponding CLT-based confidence interval for μ_1-μ_2 is

$$\mathrm{LCL} = (\bar{x} - \bar{y}) - z_{\frac{\alpha}{2}}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \text{ and } \mathrm{UCL} = (\bar{x} - \bar{y}) + z_{\frac{\alpha}{2}}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

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When population variances are equal, a confidence interval based on the z-distribution/t-distribution with pooled variance is appropriate.

Welch-Satterthwaite (WS) Method

Welch (1938) and Satterthwaite (1946) proposed confidence intervals of the difference between two means for non-normal and unequal variances situations. They proposed two intervals, given as follows:

When underlying distribution is normal

The interval is defined as follows:

$$\mathrm{LCL} = (\bar{x} - \bar{y}) - t_{df}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \text{ and } \mathrm{UCL} = (\bar{x} - \bar{y}) + t_{df}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where $df = \frac{(w_1+w_2)^2}{\left(\frac{w_1^2}{n_1-1}\right)+\left(\frac{w_2^2}{n_2-1}\right)}$, $w_1 = \frac{s_1^2}{n_1}$ and $w_2 = \frac{s_2^2}{n_2}$.

When underlying distribution is not normal

Literatures (Reed & Strak, 1996; Cressie & Whitford, 1986) show that the coverage probability for the WS interval can be much lower than its nominal level if samples come from skewed distributions. For skewed distributions, taking logarithm usually makes the distribution more symmetric. If the underlying distribution is not symmetric, apply the WS CI to the log-transformed data and finally adjust the interval to its original scale. A brief description of the procedure is as follows:

(a) Transform x_i to $\log(x_i+c_x)$ and y_i to $\log(y_i+c_y)$, where c_x and c_y are constants to make sure $x_i+c_x>0$ and $y_i+c_y>0$.

(b) Apply the WS interval to log-transformed data.

Let [Llog, Ulog] be the WS CI obtained from $\log(x_1+c_x)$, $\log(x_2+c_x)$, …, $\log(x_{n1}+c_x)$ and $\log(y_1+c_y)$, $\log(y_2+c_y)$, …, $\log(y_{n2}+c_y)$. The proposed $100(1-\alpha)\%$ CI for $\mu_1-\mu_2$ is

$$\mathrm{LCL} = \bar{y}\left(e^{\mathrm{Llog}} - 1\right) + \left(c_y e^{\mathrm{Llog}} - c_x\right) \text{ and } \mathrm{UCL} = \bar{y}\left(e^{\mathrm{Ulog}} - 1\right) + \left(c_y e^{\mathrm{Ulog}} - c_x\right)$$

Zhou and Dinh Method

Zhou and Dinh (2005) modified two-sample t-statistic to obtain better coverage when observations come from skewed distributions. The $100(1-\alpha)\%$ CI for the difference $\mu_1-\mu_2$ is given by

$$\mathrm{LCL} = (\bar{x} - \bar{y}) - N^{\frac{1}{2}}T_i^{-1}\left(N^{-\frac{1}{2}}\varepsilon_{\frac{\alpha}{2}}\right)\hat{\sigma} \text{ and } \mathrm{UCL} = (\bar{x} - \bar{y}) + N^{\frac{1}{2}}T_i^{-1}\left(N^{-\frac{1}{2}}\varepsilon_{\frac{\alpha}{2}}\right)\hat{\sigma}$$

Where $N = n_1 + n_2$

$$T_1^{-1}(t) = (\tfrac{1}{3}\widehat{A})^{-1}\left\{1 + 3\tfrac{1}{3}\widehat{A}(t - N^{-1}\tfrac{1}{6}\widehat{A})\right\}^{\frac{1}{3}} - (\tfrac{1}{3}\widehat{A})^{-1}$$

$$T_2^{-1}(t) = \left(2\tfrac{1}{3}N^{-\frac{1}{2}}\widehat{A}\right)^{-1} \log\left\{2\tfrac{1}{3}N^{-\frac{1}{2}}\,\widehat{A}\left(t - N^{-1}\tfrac{1}{6}\widehat{A}\right) + 1\right\}$$

$$T_3^{-1}(t) = \left\{1 + 3(t - N^{-1}\tfrac{1}{6}\widehat{A})\right\}^{\frac{1}{3}} - 1$$

$$\widehat{A} = \frac{(\frac{N}{n_1})^2 s_1^3 \hat{\gamma}_1 - (\frac{N}{n_2})^2 s_2^3 \hat{\gamma}_2}{\left\{\left(\frac{N}{n_1}\right)s_1^2 + (\frac{N}{n_2})s_2^2\right\}^{3/2}}$$

$$\gamma_i = \frac{n_i}{(n_i-1)(n_i-2)}\sum_{j=1}^{n_i}\left\{\frac{Y_{ij} - \bar{Y}_i}{S_i}\right\}^3, \; i = 1,2$$

$\varepsilon_\varepsilon = \varphi(\alpha)$ is the CDF of the standard normal distribution and $\widehat{\sigma} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$.

Johnson Method

Johnson (1978) proposed using the first few terms of the inverse Cornish-Fisher expansion of the t-statistic. The modified 100(1-α)% CI for μ_1-μ_2 takes the following form

$$LCL = (\bar{x} - \bar{y}) + \left(\frac{\tilde{\mu}_3}{6N\tilde{\sigma}^2}\right) + \left(\frac{\tilde{\mu}_3}{3\tilde{\sigma}^4}\right)(\bar{x} - \bar{y})^2 - t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$UCL = (\bar{x} - \bar{y}) + \left(\frac{\tilde{\mu}_3}{6N\tilde{\sigma}^2}\right) + \left(\frac{\tilde{\mu}_3}{3\tilde{\sigma}^4}\right)(\bar{x} - \bar{y})^2 + t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Where $\tilde{\mu}_3 = \left(\frac{\tilde{\mu}_{31}}{n_1}\right) - \left(\frac{\tilde{\mu}_{32}}{n_2}\right)$ and $\tilde{\mu}_3$ is an estimate of the third central moment of the ith populations.

Hall Method

In the presence of positive skewness with Hall (1992) transformation, one gets Johnson CI

$$LCL = (\bar{x} - \bar{y}) + \left(\frac{\tilde{\mu}_3}{6N\tilde{\sigma}^2}\right) + \left(\frac{\tilde{\mu}_3}{3\tilde{\sigma}^4}\right)(\bar{x} - \bar{y})^2 + \left(\frac{\tilde{\mu}_3^2}{27\tilde{\sigma}^8}\right)(\bar{x} - \bar{y})^3 - t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$UCL = (\bar{x} - \bar{y}) + \left(\frac{\tilde{\mu}_3}{6N\tilde{\sigma}^2}\right) + \left(\frac{\tilde{\mu}_3}{3\tilde{\sigma}^4}\right)(\bar{x} - \bar{y})^2 + \left(\frac{\tilde{\mu}_3^2}{27\tilde{\sigma}^8}\right)(\bar{x} - \bar{y})^3 + t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

SIMULATION STUDY

Since a theoretical comparison is not possible, a simulation study has been conducted to compare the performances of the interval intervals in this section.

Simulation Technique

The performance of the intervals is assessed in terms of their attainment of the nominal error rate a and the symmetry of the lower and upper error rates. The widely accepted level, a=0.05 is used in all our simulations. To compute the intervals, we generated data from the following parent distributions:

(1) Symmetric equal distributions: Both samples are generated from N(10,3).

(2) Symmetric unequal distributions: Two samples are generated from (N(5,3) and N(8,5)).

(3) Asymmetric with the same shape distributions: Both samples are generated from χ_2^2.

(4) Asymmetric with the different shape distributions: χ_1^2 and χ_3^2.

The sample size combinations are used (5, 10), (20, 20), (30, 40), (50, 50) and (100, 100). In each case, 5000 samples are generated and for each samples. In order to compare the performance of the various intervals, the following criteria are considered: coverage probabilities (lower, cover and upper) and mean widths of the resulting confidence intervals. The lower (upper) error probabilities of a confidence interval are calculated as the fraction out of 5000 samples that resulted in an interval that lies entirely below (above) the true value of the parameter. The coverage probability is found as the sum of the lower rate and upper rate and then subtracted from total probability 1. It is well known that if n is large, the coverage probability will be exact or close to 1-a. So the coverage probability is a useful criterion for evaluating the confidence interval. Another criterion is the width of the confidence interval. A smaller width gives a better confidence interval. It is obvious that when coverage probability is same, a smaller width indicates that the method is appropriate for the specific sample.

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RESULTS DISCUSSION

To compare the performance of the estimators, first we generated random samples from two symmetric equal distributions. Here we considered DGP as N(10,3) for both populations. The simulated results are reported in Table I. From this we see that for all sample sizes, all proposed estimators except ZD1 and ZD3 attained the nominal level 0.95 and the symmetry of lower and upper error rates. Among ZD1 and ZD3 estimators, we noticed the ZD1 estimator performed poorly in the sense of attainment of the nominal size 0.95. In Table II, we have reported performances of the considered estimators when random samples are generated from two symmetric unequal distributions i.e. N(5,3) and N(8,5). As compare to Table I, we observe better performances for the attainment of nominal confidence level for all proposed estimators. However, ZD1 and ZD3 estimators have poor coverage probability. It is noted that estimators in Table II have higher coverage probabilities than the performances listed in Table I. This is because, Table II has the wider widths compared to Table I. Our next plan is to observe how our proposed estimators perform when both are from skewed populations. In this regard, we generated random samples (with unequal sample sizes) from skewed distribution with a range of skewness, ranging 1.63–2.83 and are reported them in Tables III-IV. In Table III, we reported performances of the estimators when sampling are from asymmetric equal distributions, namely χ_2^2 with skewness 2.0. We found that ordinary t, WS1, WS2, John and ZD2 have good coverage probability. ZD1 and ZD3 coverage probability increasing as size increases and have better performances as compare to the symmetric distributions (Table I and Table II). Hall statistic suffers coverage probability deficiency as compare to symmetric distributions. In Table IV, we reported performances of the estimator in the presence of highly skewed populations. In this respect, we generated random samples from χ_1^2 with skewness 2.8284 and χ_3^2 with skewness 1.6330. We observed that t, WS1, John are performing better than WS2, ZD1, ZD2, ZD3 and Hall in the sense of attained nominal level. WS2, Hall and ZD3 are performing worse compare to the rest.

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APPLICATIONS - REAL LIFE EXAMPLES

In this section we consider two real life examples to test the performance of the considered estimators in case real life situations.

Example 1

Suppose the manufacturer of a compact disc player wanted to know whether a 10 percent reduction in price is enough to increase the sales of their product (Lind et al., 2002). To investigate, the owner randomly selected eight outlets and sold the disc player at the reduced price. At seven randomly selected outlets, the disc player was sold at the regular price. Reported below is the number of units sold last month at the sampled outlets.

Regular price:	138, 121, 88, 115, 141, 125, 96
Reduced price:	128, 134, 152, 135, 114, 106, 112, 120

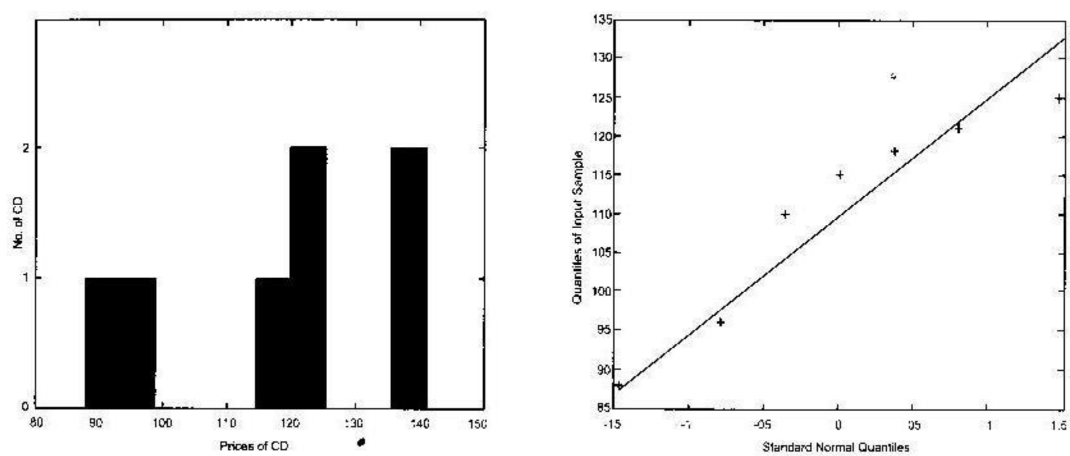

Figure I - Histogram and QQ plot of regular price of compact disc

The summary statistics of regular price data are as follows: mean =117.7143, median =121 and skewness = -0.3442. From these summary statistics and the histogram and Q-Q plots in Figure I, we may conclude that the regular price data are not normally distributed. The summary statistics of the reduced price data are as follows: mean =125.1250, median= 124 and skewness =0.4587. From these summary statistics and the histogram and Q-Q plots in Figure II, we may conclude that the reduced price data are positively skewed.

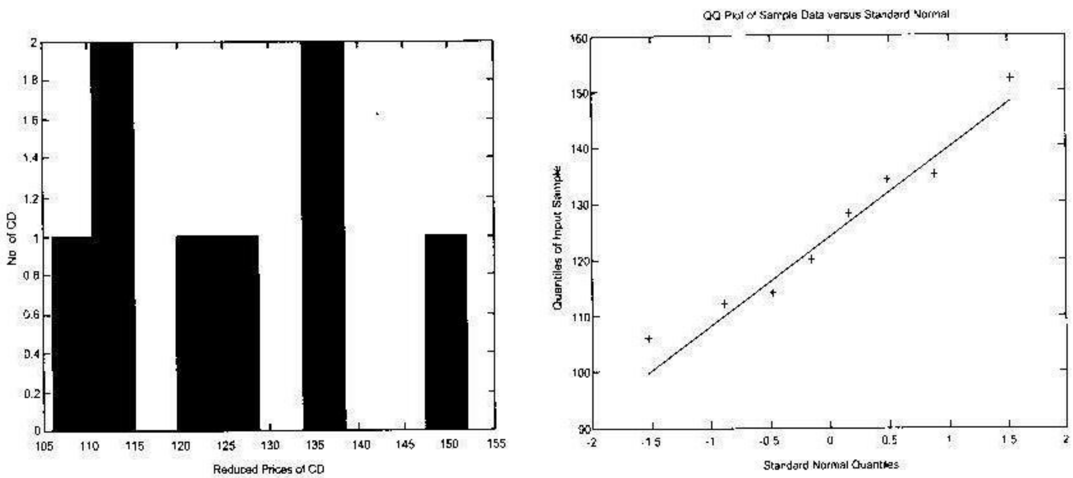

Figure II - Histogram and QQ plot of reduced price of compact disc

To verify the price reduction resulted in an increase in sales, the 95% confidence intervals for the mean difference of the regular price data and the reduced price data and the corresponding widths for the proposed estimators are given in the Table V. From this table, we observed that all proposed intervals indicated that the population means are different. However, ZD3 has the shortest width, followed by ZD2, John, t, Hall, WS1, WS2 and ZD1 respectively.

Table V: The 95% Confidence Intervals and Widths for the Mean Differences of Regular Price Data and Reduced Price Data

Estimators	**Confidence interval**	**Width**
t-interval	-12.1389 26.9603	39.0992
WS1	-12.4767 27.2981	39.7749
WS2	-12.8871 34.4585	47.3456
ZD1	-91.1051 105.9266	197.0317
ZD2	-12.0959 26.9174	39.0133
ZD3	-5.1839 20.0054	25.1893
John	-12.2754 26.8238	39.0992
Hall	-12.2787 26.8205	39.0992

[redacted]

Example 2

Suppose the commercial Bank and Trust Company is studying the use of its automatic teller machines (ATMs) (Lind et al., 2002). Of particular interest is whether young adults (under 25 years) use the machines more than senior citizens. To investigate further, samples of customers under 25 years of age and customers over 60 years of age were selected. The number of ATM transactions last month was determined for each selected individual and the results are shown below:

Under 25:	10, 10, 11, 15, 7, 11, 10, 9
Over 60:	4, 8, 7, 7, 4, 5, 1, 7, 4, 10, 5

The summary statistics of under 25 data are as follows: mean = 10.3750, median = 10 and skewness = 0.7663. From these summary statistics and the histogram and Q-Q plots in Figure III, we may conclude that the under 25 data is positively skewed. The summary statistics of the over 60 data are as follows: mean = 5.6364, median = 5 and skewness = -0.0663. From these summary statistics and the histogram and Q-Q plots in Figure IV, we may conclude that the over 60 group data is skewed.

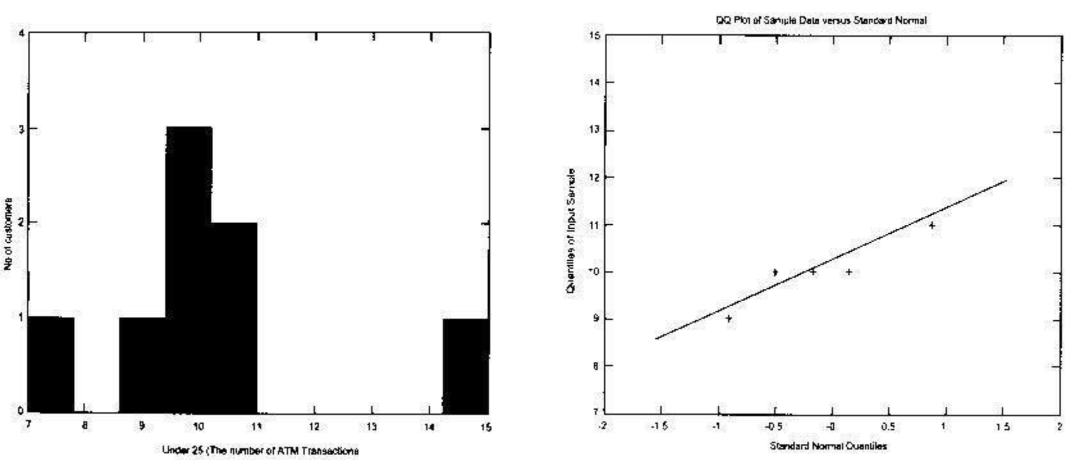

Figure III - Histogram and QQ plot of under 25

The 95% confidence intervals for the difference between the two means of the under 25 and over 60 are given in Table VI. From this table, we observed that all proposed intervals showed that the population means are different. However, WS2 has the shortest width, followed by ZD3, ZD2, John, t, Hall, WS1 and ZD1 respectively.

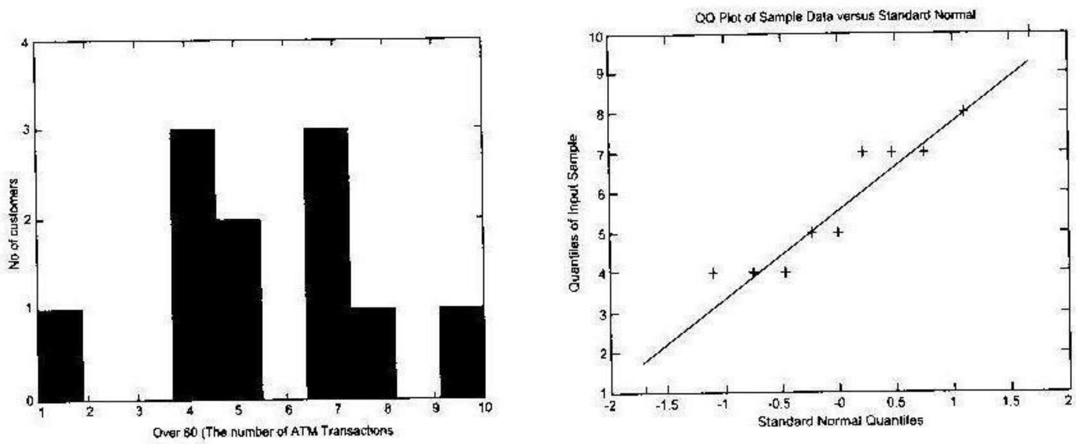

Figure IV - Histogram and QQ plot of over 60

Table VI: The 95% Confidence Intervals and Widths for Mean Differences of Under 25 and Over 60

Estimators	Confidence interval	Width
t-interval	-7.0734 -2.4039	4.6695
WS1	-7.0851 -2.3922	4.6930
WS2	-4.3128 -1.6928	2.6201
ZD1	-10.2452 0.7679	11.0131
ZD2	-6.7383 -2.7390	3.9993
ZD3	-6.2940 -3.1832	3.1108
John	-7.0566 -2.3871	4.6695
Hall	-7.0271 -2.3576	4.6695

CONCLUSION

In this paper some estimators for estimating the differences between two skewed population means are considered. A simulation study has been conducted to compare the performance of the proposed interval estimators. Two real life business examples have been analyzed to illustrate the application of the considered confidence intervals. We considered the following estimators, namely, the ordinary-t, two versions of the Welch-Satterthwaite method, three versions of the Zhou and Dinh method, the Johnson method and the Hall method. Conclusions have been made based on coverage probability and width of the intervals. Our findings indicated that the considered estimators have good coverage probability and shortest width for symmetric and also for skewed distributions. In the presence of moderate and skewed populations, Johnson, Hall, and ZD can be chosen. In the presence of symmetric distribution, the ordinary-t, WS estimators can be chosen.

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APPENDIX

Table I: Coverage Probabilities when DGP from Symmetric Equal Distributions: N(10,3) with Skewness 0.0

Rate	**t**	**WS1**	**WS2**	**ZD1**	**ZD2**	**ZD3**	**John**	**Hall**
				n_1=5 and n_2=10				
Lower	0.0330	0.0272	0.0188	0.2704	0.0846	0.1802	0.0330	0.0332
Cover	0.9388	0.9502	0.9550	0.4576	0.8982	0.8056	0.9384	0.9374
Upper	0.0282	0.0226	0.0262	0.2720	0.0172	0.0142	0.0286	0.0294
MW	6.8404	7.3606	8.7664	16.274	6.4618	4.3685	6.8404	6.8404
				n_1=20 and n_2=20				
Lower	0.0244	0.0244	0.0232	0.2620	0.0472	0.0388	0.0248	0.0244
Cover	0.9520	0.9520	0.9532	0.4884	0.9330	0.9612	0.9518	0.9514
Upper	0.0236	0.0236	0.0236	0.2496	0.0198	0.0000	0.0234	0.0242
MW	3.8167	3.8233	4.5699	12.463	3.7003	2.8889	3.8167	3.8167
				n_1=30 and n_2=40				
Lower	0.0216	0.0216	0.0174	0.2446	0.0356	0.0214	0.0220	0.0224
Cover	0.9520	0.9522	0.9546	0.5146	0.9450	0.9786	0.9514	0.9516
Upper	0.0264	0.0262	0.0280	0.2408	0.0194	0.0000	0.0266	0.0260
MW	2.8809	2.8867	3.4808	9.3346	2.8316	2.3196	2.8809	2.8809
				n_1=50 and n_2=50				
Lower	0.0258	0.0258	0.0260	0.2316	0.0336	0.0068	0.0258	0.0248
Cover	0.9488	0.9488	0.9510	0.5516	0.9448	0.9932	0.9488	0.9496
Upper	0.0254	0.0254	0.0230	0.2168	0.0216	0.0000	0.0254	0.0256
MW	2.3714	2.3720	2.8532	7.6389	2.3425	1.9703	2.3714	2.3714
				n_1=100 and n_2 =100				
Lower	0.0242	0.0242	0.0234	0.1822	0.0326	0.0010	0.0242	0.0238
Cover	0.9472	0.9472	0.9514	0.6460	0.9458	0.9990	0.9472	0.9462
Upper	0.0286	0.0286	0.0252	0.1718	0.0216	0.0000	0.0286	0.0300
MW	1.6691	1.6692	2.0223	5.3853	1.6590	1.4552	1.6691	1.6691

Note: t – Ordinary t, WS1- Welch-Satterthwaite, ZD1 – Zhou and Dinh 1, ZD2 – Zhou and Dinh 2, ZD3 – Zhou and Dinh 3, John- Johnson, Hall – Hallm MW – Mean Width

Table II: Coverage Probabilities when DGP from Symmetric Unequal Distributions: N(5,3) and N(8,5) with Skewness 0.0

Rate	**t**	**WS1**	**WS2**	**ZD1**	**ZD2**	**ZD3**	**John**	**Hall**
				n_1=5 and n_2=10				
Lower	0.0254	0.0244	0.0074	0.2638	0.0680	0.1280	0.0250	0.0254
Cover	0.9502	0.9518	0.9742	0.4902	0.9184	0.8672	0.9496	0.9496
Upper	0.0244	0.0238	0.0184	0.2460	0.0136	0.0048	0.0254	0.0250
MW	8.8156	9.0228	17.782	23.626	8.1964	5.6321	8.8156	8.8156
				n_1=20 and n_2=20				
Lower	0.0236	0.0230	0.0066	0.2410	0.0484	0.0546	0.0238	0.0232
Cover	0.9506	0.9516	0.9862	0.5122	0.9334	0.9448	0.9504	0.9502
Upper	0.0258	0.0254	0.0072	0.2468	0.0182	0.0006	0.0258	0.0266
MW	5.2213	5.2633	7.6845	17.184	5.0664	3.9522	5.2213	5.2213
				n_1=30 and n_2=40				
Lower	0.0230	0.0230	0.0034	0.2412	0.0336	0.0140	0.0230	0.0224
Cover	0.9534	0.9534	0.9904	0.5178	0.9450	0.9860	0.9534	0.9538
Upper	0.0236	0.0236	0.0062	0.2410	0.0214	0.0000	0.0236	0.0238
MW	3.8291	3.8335	5.5893	12.345	3.7614	3.0827	3.8291	3.8291
				n_1=50 and n_2=50				
Lower	0.0244	0.0244	0.0072	0.2136	0.0382	0.0148	0.0244	0.0244
Cover	0.9516	0.9522	0.9862	0.5634	0.9452	0.9852	0.9518	0.9522
Upper	0.0240	0.0234	0.0066	0.2230	0.0166	0.0000	0.0238	0.0234
MW	3.2599	3.2693	4.5619	10.542	3.2206	2.7086	3.2599	3.2599
				n_1=100 and n_2 =100				
Lower	0.0228	0.0228	0.0056	0.1916	0.0270	0.0040	0.0228	0.0224
Cover	0.9524	0.9526	0.9862	0.6260	0.9506	0.9960	0.9524	0.9522
Upper	0.0248	0.0246	0.0082	0.1824	0.0224	0.0000	0.0248	0.0254
MW	2.2969	2.3001	3.1759	7.4192	2.2830	2.0026	2.2969	2.2969

See Footnote of Table I

Table III: Coverage Probabilities when DGP from Chi-square Distribution with 2 df with Skewness 2.00

Rate	t	WS1	WS2	ZD1	ZD2	ZD3	John	Hall
				n_1=5 and n_2=10				
Lower	0.0452	0.0352	0.0174	0.1960	0.0538	0.2286	0.0488	0.0126
Cover	0.9408	0.9560	0.9536	0.6094	0.9318	0.6822	0.9376	0.8686
Upper	0.0140	0.0088	0.0290	0.1946	0.0144	0.0892	0.0136	0.1188
MW	4.3665	4.7583	10.611	8.3546	3.9910	2.7660	4.3665	4.3665
				n_1=20 and n_2=20				
Lower	0.0230	0.0230	0.0222	0.1798	0.0120	0.2228	0.0242	0.0208
Cover	0.9528	0.9528	0.9562	0.6354	0.9756	0.7316	0.9506	0.9072
Upper	0.0242	0.0242	0.0216	0.1848	0.0124	0.0456	0.0252	0.0720
MW	2.5052	2.5167	3.7560	6.9731	2.4517	1.8963	2.5052	2.5052
				n_1=30 and n_2=40				
Lower	0.0268	0.0268	0.0208	0.1852	0.0118	0.1216	0.0292	0.0234
Cover	0.9498	0.9500	0.9498	0.6706	0.9732	0.8190	0.9468	0.9222
Upper	0.0234	0.0232	0.0294	0.1442	0.0150	0.0594	0.0240	0.0544
MW	1.9004	1.9062	2.6651	5.7003	1.8390	1.5245	1.9004	1.9004
				n_1=50 and n_2=50				
Lower	0.0258	0.0258	0.0246	0.2020	0.0104	0.1888	0.0266	0.0596
Cover	0.9496	0.9496	0.9486	0.6852	0.9730	0.7844	0.9478	0.9282
Upper	0.0246	0.0246	0.0268	0.1128	0.0166	0.0268	0.0256	0.0122
MW	1.5726	1.5740	2.1487	5.1798	1.5566	1.3069	1.5726	1.5726
				n_1=100 and n_2 =100				
Lower	0.0232	0.0232	0.0284	0.1604	0.0070	0.1366	0.0236	0.0246
Cover	0.9524	0.9524	0.9448	0.6916	0.9698	0.8486	0.9510	0.9328
Upper	0.0244	0.0244	0.0268	0.1480	0.0232	0.0148	0.0254	0.0426
MW	1.1076	1.1078	1.4657	3.6801	1.1012	0.9657	1.1076	1.1076

See Footnote of Table I

Table IV: Coverage Probabilities when DGP from Chi-square Distribution with 1 df with Skewness 2.8284 and 3 df with Skewness 1.6330

Rate	**t**	**WS1**	**WS2**	**ZD1**	**ZD2**	**ZD3**	**John**	**Hall**
				n_1=5 and n_2=10				
Lower	0.0182	0.0180	0.4582	0.1600	0.0462	0.5334	0.0194	0.0052
Cover	0.9554	0.9566	0.5418	0.7026	0.9438	0.4646	0.9478	0.8434
Upper	0.0264	0.0254	0.0000	0.1374	0.0100	0.0020	0.0328	0.1514
MW	4.0936	4.2615	8.5200	7.1477	4.2473	2.6195	4.0936	4.0936
				n_1=20 and n_2=20				
Lower	0.0170	0.0164	0.3342	0.1416	0.0436	0.5024	0.0174	0.0146
Cover	0.9450	0.9468	0.6658	0.7030	0.9420	0.4892	0.9442	0.9180
Upper	0.0380	0.0368	0.0000	0.1554	0.0144	0.0084	0.0384	0.0674
MW	2.4969	2.5263	1.1178	6.4269	2.5936	1.9067	2.4969	2.4969
				n_1=30 and n_2=40				
Lower	0.0158	0.0158	0.2400	0.1316	0.1416	0.4494	0.0178	0.0142
Cover	0.9566	0.9566	0.7600	0.7222	0.7030	0.5302	0.9546	0.9330
Upper	0.0276	0.0276	0.0000	0.1462	0.1554	0.0204	0.0276	0.0528
MW	1.8288	1.8338	0.8105	6.1985	6.1985	1.8401	1.8288	1.8288
				n_1=50 and n_2=50				
Lower	0.0190	0.0186	0.0892	0.1302	0.0366	0.3136	0.0204	0.0198
Cover	0.9440	0.9450	0.8968	0.7330	0.9476	0.6238	0.9428	0.9340
Upper	0.0370	0.0364	0.0140	0.1368	0.0158	0.0626	0.0368	0.0462
MW	1.5781	1.5840	0.6067	5.8311	1.6062	1.3199	1.5781	1.5781
				n_1=100 and n_2 =100				
Lower	0.0188	0.0188	0.0212	0.1348	0.0354	0.0234	0.0190	0.0224
Cover	0.9478	0.9478	0.9788	0.7460	0.9520	0.9652	0.9542	0.9502
Upper	0.0334	0.0334	0.0000	0.1192	0.0126	0.0114	0.0268	0.0274
MW	1.1093	1.1112	0.4045	4.4055	1.1213	0.9722	1.1093	1.1093

See Footnote of Table I